UDC 539.3

ON THE FREQUENCY SPECTRA OF NATURAL VIBRATIONS OF SHELLS OF REVOLUTION

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An asymptotic solution of the problem of determining the frequency spectrum of natural vibrations of shells of revolution is given. The order of the error hence admitted is estimated in an example of evaluating the zeroes of Bessel functions. The question of the density of the frequency distribution is discussed. A comparison with the results obtained in [1], as well as with an empirical frequency density, is given in the example of cylindrical and closed spherical shells.

1. Let a shell of revolution vibrate with the frequency ω. Then the vibration mode can be found from the equations given in [2], which are, when tangential inertial forces are neglected,

$$D\Delta\Delta\omega - \Delta_k\psi - \rho h\omega^2 w = 0, \qquad (Eh)^{-1}\,\Delta\Delta\psi + \Delta_k w = 0 \tag{1.1}$$

Here w is the normal deflection function, and ψ the stress resultant function in the middle surface. Selecting the arclength x along a generator and the angle θ measured in a circumferential direction as coordinates, we arrive at expressions for the operators used in (1.1)

$$\Delta = \frac{1}{r}\frac{\partial}{\partial x}\left(r\frac{\partial}{\partial x}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \qquad \Delta_k = \frac{1}{r}\frac{\partial}{\partial x}\left(\frac{r}{R_2}\frac{\partial}{\partial x}\right) + \frac{1}{rR_1}\frac{\partial^2}{\partial\theta^2}$$

where $r(x)$ is the distance to the axis of the shell of revolution, and R_1, R_2 are the radii of curvature of the shell.

Let us seek the solution of (1.1) by extracting the rapidly varying factors [3, 4]

$$\omega(x,\theta) = W(x)\exp[if(x)]\cos n\theta, \quad \psi(x,\theta) = \Psi(x)\exp[if(x)]\cos n\theta \tag{1.2}$$

The functions $W(x)$, $\Psi(x)$ and the derivative $df/dx = f'(x)$ will be considered to vary slowly, so that the components W', Ψ', f'' can be neglected as compared with $f'W$, $f'\Psi$, $(f')^2$. Substituting (1.2) into (1.1) we see that if the function $f'(x)$ is known, the natural vibrations frequency can be found from the expression

$$\rho h\omega^2 = D\left(f'^2 + \frac{n^2}{r^2}\right)^2 + Eh\left(\frac{f'^2}{R_2} + \frac{n^2}{r^2R_1}\right)^2\left(f'^2 + \frac{n^2}{r^2}\right)^{-2} \tag{1.3}$$

The quantities f' and n/r are analogs of the wave numbers k_1, k_2 used in [1]. For a known frequency expression (1.3) should be treated as an equation to determine f'. Let us study this equation in more detail.

2. For some frequency let (1.3) have no real roots for part of the shell. The deformation of this part of the shell, later called a zero-type zone, is almost quasi-static. This part of the shell plays the part of an elastic frame for the rest of the shell. The solutions in the zero-type zone will be of the edge effect type. From the structure of (1.3) there results that two kinds of vibrating zones of the shell are possible. The first type is characterized by there being only one positive real root. Characteristic for the zone of the second type is the presence of two real positive roots; the case of degeneration of the dynamic edge effect [5] holds. We obtain the zone boundary by setting $f' = 0$, which yields

$$\rho h \omega^2 = D\frac{n^4}{r^4} + Eh\frac{1}{R_1^2} \tag{2.1}$$

We find the other boundaries by assuming that a real positive root of multiplicity two exists. The equation of the boundary is

$$a^2(z^2 + 12Kc^2) + (z^3 - 4Kc^2z + 18Ka^2c^2)(2z^2 - 8Kc^2 - 3a^2z) = 0$$

$$ac(z^2 + 12Kc^2)(z^3 - 4Kc^2z + 18Ka^2c^2)^{-1} + 1 < 0 \tag{2.2}$$

$$z = \frac{R^2}{R_2^2} - \frac{\rho h\omega^2 R^2}{Eh}, \quad a = \frac{R}{R_2}, \quad c = \frac{R}{R_1} - \frac{R}{R_2}, \quad K = \frac{DR^2n^4}{Ehr^4}$$

The characteristic radius R can be taken constant. The lines along which the inequality in (2.2) is not satisfied are not the boundaries.

If the number n is small so that the strong inequality $Dn^4r^{-4} \ll EhR_1^{-2}$ is not satisfied, then the degenerate equation of [6] (equation (1.3) without the first member on the right side) can be used to determine the roots in the zone of the second type. The lines (2.1) can be boundaries between zones of zero-type and zones of the first type, as well as between zones of the first and second types. The lines (2.2) are boundaries between zones of the second and zero types. Examples of the partitioning of the middle

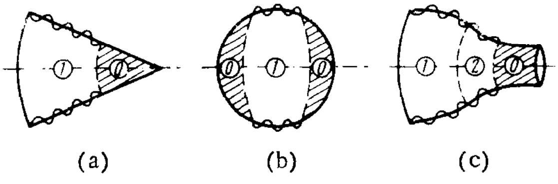

Fig. 1.

surface of shells of revolution are presented in Fig. 1.

3. The approximate determination of the natural vibration frequencies will be distinct, depending on the manner of alternation of the zones of zero, first and second types.

Let us first examine the simplest case when there are no zones of second type (Fig. 1a, b). The function $f(x)$ acquires some increment in each of the zones of first type. Let us consider the vibrations to originate only when this increment is a multiple of π. If there are several zones of the first type, then as a rule, vibrations will be observed in only one of these zones at a certain natural frequency.

Let us consider the general case when there are zones of all three types (Fig. 1c). Let a zone of second type be surrounded by zones of zero type. In this case, only an upper bound can be estimated for the number of natural frequencies. For the upper bound let us consider the natural frequencies to correspond to increments of one of the functions $f(x)$ which are multiples of π. In reality, part of the frequencies found may not be realized. If a zone of the second type bounds at least one side of a zone of the first type, then all the frequencies are realized. For example, let a zone of the second type be the boundary of zones of zero and first types, and let a zone of zero type follow a zone of first type. In this case, two kinds of vibrations are possible. The first kind is associated with the lesser root and corresponds to vibrations just in the second zone. The increment of the appropriate function $f(x)$ should be a multiple of π. The second kind is associated with the highest root and corresponds to vibrations in the first and second zones. Analogously, when zones of the zeroth, second, first, second and zeroth types alternate, vibrations in each of the zones of the second type, as well as simultaneous vibrations in all three zones are possible.

To estimate the error admitted, let us consider the problem of calculating the roots of the Bessel function $I_n(y)$. The equation has the form

$$\frac{d^2w}{dx^2}+\frac{1}{x}\frac{dw}{dx}+\left(y^2-\frac{n^2}{x^2}\right)w=0 \tag{3.1}$$

Examining the solution of (3.1) in the segment [0, 1] under the condition of boundedness as $x \to 0$ and the condition $w = 0$ for $x = 1$, we obtain the exact equation $I_n(y) = 0$ and the approximate equation

$$n\left[(y^2n^{-2}-1)^{1/2}-\arcsin\,(1-n^2y^{-2})^{1/2}\right]=m\pi \tag{3.2}$$

The results of calculating the roots of (3.2) are presented in Fig. 2 as dashed lines. The exact dependencies are given by the solid lines. A comparison shows that the error of the approximate formula does not exceed π for all values of m and n.

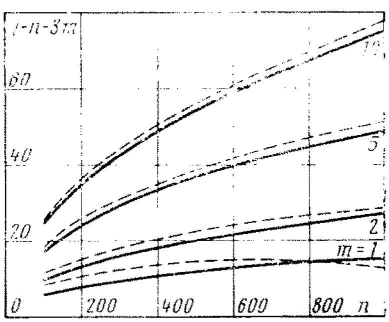

Fig. 2.

4. As illustrations, let us consider the vibrations of cylindrical and closed spherical shells.

Two cases are possible for a circular cylindrical shell ($R_1 = \infty$, $R_2 = R$) of length l In the first case the whole shell is a zone of zero type and in the second a zone of the first type. The solution of (1.3) for f' is independent of the coordinates. From the condition that the increment $f(x)$ must be a multiple of the number π, we obtain $f' = \pi m l^{-1}$. For the frequencies we find the expression

$$\omega = \omega_0 f(m, n)$$

Here

$$f(m,n) = [(\gamma^2m^2+n^2)^2+m_*^4\gamma^4m^4(\gamma^2m^2+n^2)^{-2}]^{1/2}$$

$$\gamma=\pi Rl^{-1}, \qquad \omega_0=(D/\rho hR^4)^{1/2}, \qquad m_*=(EhR^2/D)^{1/4}$$

The number of frequencies $N(\omega)$ less than ω and the density of the frequencies $n(\omega) = dN/d\omega$ are given by the approximate formulas

$$N(\omega)=\iint_S dm\,dn \qquad (S: f(m,n)<\omega/\omega_0)$$

$$\omega_0 n(\omega)=\begin{cases}\dfrac{2}{\gamma}\left(\dfrac{\omega}{\omega_0}\right)^{1/2}K\left[\left(\dfrac{2\omega_*}{\omega+\omega_*}\right)^{1/2}\right] & (\omega<\omega_*)\\[2ex] \dfrac{2}{\gamma}\left(\dfrac{\omega}{2\omega_*}\right)^{1/2}K\left[\left(\dfrac{\omega+\omega_*}{2\omega_*}\right)^{1/2}\right] & (\omega>\omega_*)\end{cases} \tag{4.1}$$

Here $\omega_* = \omega_0 m_*^2$. Formula (4.1) agrees with the corresponding formula obtained in [1]. To compute the multiplicity it is sufficient to double the densities found. For $\omega = \omega_*$ there is a point of frequency concentration. The frequency ω_* equals the frequency of natural axisymmetric membrane vibrations. Singularities in the densities of the natural frequency distribution of thin shells were first mentioned in [1].

In the case of a closed spherical shell $R_1 = R_2 = R$ for $n \neq 0$ there are zones of zeroth type near the poles, which bound a zone of the first type. The equation to determine $f'(x)$ turns out to be a biquadratic. The solution of this equation is

$$f' = \left[\left(\frac{\rho h\omega^2}{D} - \frac{Eh}{R^2}\right)^{1/2} - \frac{n^2}{r^2}\right]^{1/2}, \qquad r = R\sin\frac{x}{R}$$

From the condition $f' = 0$ we find the limits of integration

$$x_{1,2} = R\left\{\frac{\pi}{2} \pm \left[\frac{\pi}{2} - \arcsin\frac{n}{R}\left(\frac{DR^2}{\rho h^2 R^2\omega^2 - Eh}\right)^{1/4}\right]\right\}$$

The frequency equation is

$$\int_{x_1}^{x_2} f'(x)\,dx = m\pi \qquad (m = 1, 2, \ldots)$$

We hence obtain an approximate expression for the frequencies

$$\omega = \omega_0\left[(m+n)^4 + m_*^4\right]^{1/2}$$

The number of frequencies $N(\omega)$ less than ω is given by a formula taking account of the multiplicity of the frequencies

$$N(\omega) \approx 2 \iint\limits_{(m+n)<(\omega^2\omega_0^{-2} - m_*^4)^{1/4}} dm\,dn = \left(\frac{\omega^2}{\omega_0^2} - m_*^4\right)^{1/2}$$

The density of the frequency spectrum of the vibrations is

$$n(\omega) = \frac{dN(\omega)}{d\omega} \approx \frac{\omega\omega_0^{-2}}{(\omega^2\omega_0^{-2} - m_*^4)^{1/2}} \tag{4.2}$$

The frequency density equals zero for $\omega < \omega_*$ and tends to ω_0^{-1} as the frequency grows. The frequency ω_* corresponds to the membrane vibrations frequency of a spherical shell. The limit value ω_0^{-1} agrees with the frequency density for a plate of area $4\pi R^2$, i.e., the area of the whole spherical shell. Formula (4.2) can also be obtained by integrating the frequency density from [1] over the area of the shell surface.

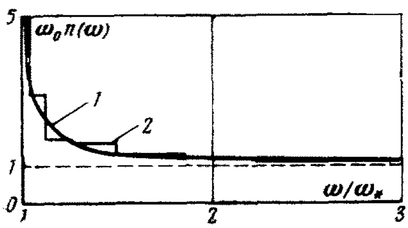

Fig. 3.

Presented in Fig. 3 are results of calculating the reduced frequency density $\omega_0 n(\omega)$. Curve 1 has been obtained by means of (4.2). Curve 2 corresponds to averaging the number of frequencies obtained by exact integration of (1.1) over the sections.

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ON THE THERMODYNAMIC INTERPRETATION OF THE EVOLUTIONARY CONDITIONS OF THE EQUATIONS OF THE MECHANICS OF FINITELY DEFORMABLE VISCOELASTIC MEDIA OF MAXWELL TYPE

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A thermodynamic interpretation is given of the phenomenon of the loss of evolutionarity in the hydrodynamics equations of viscoelastic incompressible fluids corresponding to models proposed in [1, 2].

A Clausius inequality is formulated for the virtual perturbations of the equilibrium parameters on the basis of the second law of thermodynamics and propositions on the local thermodynamic equilibrium in a small particle of a continuous medium [3].

Properties of reversible instantaneous deformations in the considered media are investigated and the form of the integral energy is found. The internal energy in the Oldroyd models [1] depends on the first invariant of the viscoelastic stress tensor, but can also be expressed in terms of the reversible strain components. In the De Witt model [2] it depends on the second invariant of the stress tensor and is nonlocal relative to the reversible strain.

Necessary conditions for the thermodynamic stability are obtained by using the expressions found for the internal energy and the Clausius inequality. Constraints on the principal values of the viscoelastic stress tensor result from these conditions which have been established earlier on the basis of demands for the evolutionarity of the corresponding systems of hydrodynamics equations [4, 5].

1. Clausius inequality as a requirement for the stability of the local thermodynamic equilibrium of an element of a continuous